The toughness of a composite containing short brittle fibres

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A method is presented whereby various potential contributions to the toughness of a polymer containing short brittle fibres can be quantified. It relies on a model for predicting the cumulative probability distribution of fibre pull-out lengths. The method reveals that toughness increases to a maximum value with increasing fibre length. Good agreement between theory and experiment supports the validity of our approach.

1. Introduction

A composite containing randomly dispersed, aligned short fibres of uniform strength will always exhibit fibre pull-out preceding fast fracture (Fig. 1). If the (frictional) shear stress, τ, at the fibre–matrix interface is maintained during the extraction of a fibre, then the average work done in pulling out a fibre whose length is l and diameter is d is approximately [1]:

$$\langle W_p \rangle = \frac{\int_0^{l/2} \pi d\tau l^2 \mathrm{d}l}{2\int_0^{l/2} \mathrm{d}l} \qquad (l < l_c)$$
$$= \pi d\tau l^2/24 \qquad (1)$$

The limit $l/2$ is chosen rather than l since a fibre is always likely to pull out from a matrix fracture surface beneath which it is least embedded (Fig. 1).

For the case where fibres are longer than a critical length†, l_c, a fraction, l_c/l, only, of the fibres are extracted, while the remainder break in the plane of the matrix crack. The average work to pull the fibre out of its socket is approximately [1]:

$$\langle W_p \rangle = \left(\frac{l_c}{l}\right)\frac{\int_0^{l_c/2} \pi d\tau l^2 \mathrm{d}l}{2\int_0^{l_c/2} \mathrm{d}l} \qquad (l > l_c)$$
$$= \left(\frac{l_c}{l}\right)\frac{\pi d\tau l_c^2}{24} \qquad (2)$$

In physical terms, the work of fibre pull-out is dependent upon the (frictional) shear stress, τ, at the interface which is related to the coefficient of friction, μ, and the pressure of the matrix on the fibre normal to its surface, for example the shrinkage of epoxy on to a carbon fibre during processing. (This, in turn, is related to the fibre–matrix misfit strain, ε_0, that appears in Equation 3a below). But of particular importance to this model, Equation 2 predicts that the work of fibre pull-out tends to zero as the length, l, of a fibre that is uniformly strong increases to infinity (Fig. 2). The inference is that the toughness of the composite can be equated to the fracture energy of the matrix, or some fraction of it, in the absence of pull-out or any other toughening mechanism. This is not universally true; for an epoxy containing long carbon fibres we measure a toughness of the order of $10^4\,\mathrm{J\,m^{-2}}$ which is two orders of magnitude greater than that of the unreinforced matrix [2]. Examination of a fracture surface of the carbon fibre composite in the scanning electron microscope reveals broken fibres of various lengths protruding above the fracture plane of the cracked matrix [2]. This is because of the variation in strength of the carbon along its length and the snapping of the fibre at one (or more) of the weaker points beneath the surface of the matrix fracture plane [2].

2. The energetics of fibre debonding, fibre fracture and fibre pull-out

The problem of identifying the dominant mechanism of fracture can be tackled in two complementary ways; either the dominance of the more easily recognisable mechanisms like fibre pull-out are determined by optical or electron microscopy [2, 3], or alternatively, a link is made between our understanding of these mechanisms and theoretical models [4, 5]. Here, the aim is to derive for each mechanism, in turn, an energy-dissipative equation which is based on a physically sound microscopic model.

Our starting point assumes a matrix crack spanned by an unbroken, partially debonded brittle fibre [4] (Fig. 3). First, we derive an equation which describes the build-up of stress, $\sigma(x)$, in the fibre from the tip of the cylindrical debond crack to the plane of the matrix fracture surface [4]:

$$\sigma(x) = \sigma_p - (\sigma_p - \sigma_d)\exp(-\beta x) \qquad (3)$$

σ_p is the maximum stress built up in the debonded fibre by friction at the interface:

$$\sigma_p = \varepsilon_0 E_f/\nu_f \qquad (3a)$$

The elastic coefficient, β, takes into account Poisson contraction of the fibre under load:

$$\beta = \frac{4\mu\nu_f E_m}{E_f d(1 + \nu_m)} \qquad (3b)$$

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† l_c is the minimum length of bonded fibre embedded in a matrix which can be broken in a monotonic tensile test.

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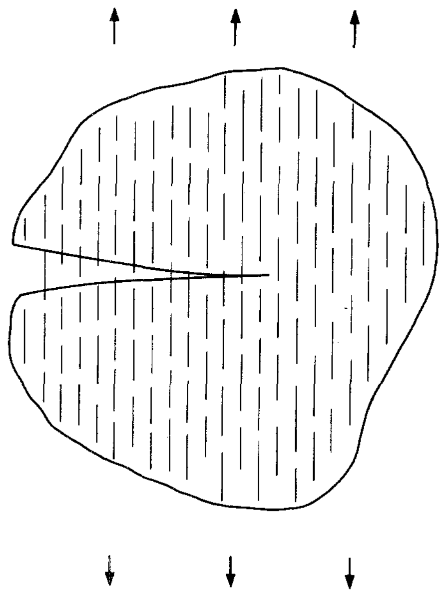

Figure 1 A schematic diagram showing aligned short fibres randomly dispersed in a cracked matrix.

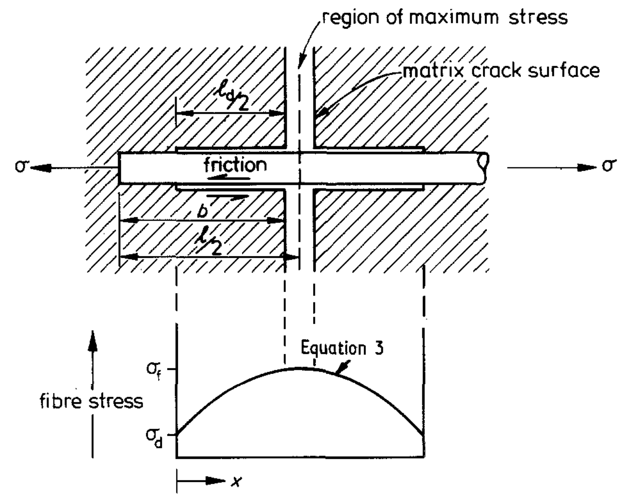

Figure 3 The coordinate system of a short, partially debonded fibre spanning a matrix crack. b is the depth of that end of fibre least buried beneath the surface of the crack plane. Also, a schematic diagram showing the stress distribution along the partially debonded length of fibre.

The fibre stress, σ_d, at the tip of the debond crack is given by

$$\sigma_d = \left(\frac{8E_f\Gamma_i}{d}\right)^{1/2} \tag{3c}$$

E_f, ν_f, E_m and ν_m are the moduli and Poisson's ratios of the fibre and matrix, respectively, and Γ_i is the fracture energy of the interface.

As the load on the fibre increases, its diameter, d, decreases (Poisson contraction), and the debond crack extends stably, reaching a maximum length only when the fibre snaps at σ_f. Rewriting Equation 3,

$$\sigma_f = \sigma_p - (\sigma_p - \sigma_d)\exp(-\beta l_d/2) \tag{4}$$

Hence

$$l_d = \frac{2}{\beta}\ln\left(\frac{\sigma_p - \sigma_d}{\sigma_p - \sigma_f}\right) \tag{5}$$

The likelihood that the fibre break is on the same plane as the matrix crack is small, even though the position of maximum fibre stress coincides with the matrix fracture surface. Macroscopic crack growth in the composite transverse to the direction of fibre alignment is accompanied, therefore, with the pulling out of the majority of fibre ends, broken or still intact.

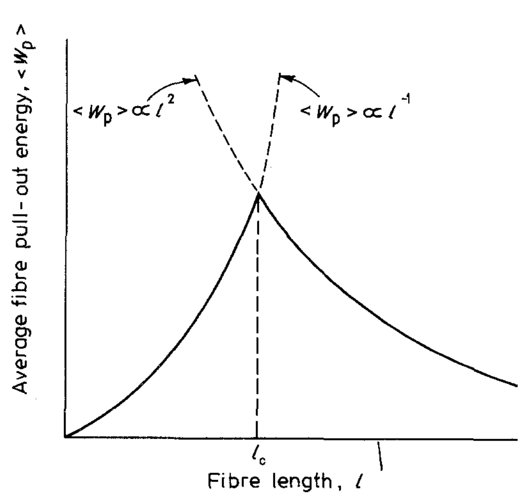

Figure 2 A schematic diagram showing the relation between fibre pull-out energy and fibre length (after Kelly [1]).

The energetics of complete fracture of the composite is based, therefore, on a sequence of events that can occur in a damage process zone in front of and in the wake of a propagating crack [4, 5]. Energy is dissipated during matrix cracking (this is small, some $300\,\mathrm{J\,m^{-2}}$ or less), decohesion (debonding) of the fibre–matrix interface (the fracture energy Γ_i is much smaller but the total surface area can be enormous), fibre breakage (the release of stored elastic strain energy in the fibre over its debonded length), and fibre pull-out (work done against friction).

2.1. Toughening by interfacial debonding

The contribution this mechanism makes to the toughness of the composite is in proportion to the total area of debonded interface [4, 5]:

$$G_d = 8l_d\Gamma_i V_f/d \quad (b > l_d/2) \tag{6}$$

where b is defined in Fig. 3 and V_f is the fibre volume fraction.

The energy associated with the debonding of a fibre can be estimated by substituting for l_d (Equation 5) into Equation 6. In a composite containing short fibres where one end of the fibre is close to the matrix crack, the maximum length of debond crack, l_d, given by Equation 5, will be unattainable if the fibre pulls out. The model (Equation 6) therefore has to be modified for the case where $b < l_d/2$ (Fig. 3):

$$G_d = 8b\Gamma_i V_f/d \quad (b < l_d/2) \tag{6a}$$

2.2. Toughening by fibre fracture

Under increasing load, the debond crack will continue to spread along the interface whose fibre length is greater than b from the matrix crack. The fibre diameter will decrease steadily until finally snapping. A consequence of Poisson contraction is that the rate of load build-up in the fibre decreases with distance from the tip of the debond crack to the plane of the matrix crack [4] (Fig. 3). When the fibre snaps, only some of its stored elastic energy is released since the load builds up once more from the broken fibre end [4, 5]. An estimation of the dissipated elastic energy can be made from a knowledge of the states of stress in the fibre and matrix immediately before and after it breaks [4, 5]:

$$G_f = \frac{V_f}{E_f}\left(\frac{\sigma_p^2 l_d}{2} - \frac{(\sigma_p - \sigma_d)^2(e^{-\beta l_d} - 1)}{2\beta} + \frac{2\sigma_p(\sigma_p - \sigma_d)(e^{-\beta l_d/2} - 1)}{\beta}\right), \quad (b > l_d/2) \tag{7}$$

The energy associated with fibre fracture can be estimated by substituting l_d (Equation 5) into Equation 7. Likewise, the model has to be modified when $b < l_d/2$, by replacing l_d in Equation 7 with b.

2.3. Toughening by fibre pull-out

Provided the lengths of protruding fibres have a uniform probability distribution, then the average work of pull-out is dependent on the maximum fibre stress during pull-out, σ_p, the average fibre pull-out length $\langle l_p \rangle$, the elastic coefficient, β, which accounts for Poisson contraction of the fibre, and the coefficient of friction at the interface, μ (which is included in β) [4]:

$$\langle G_{po} \rangle = V_f \sigma_p \left(\langle l_p \rangle + \frac{e^{-\beta\langle l_p \rangle} - 1}{\beta}\right) \tag{8}$$

The energy associated with fibre pull-out can be estimated by substituting the average pull-out length, $\langle l_p \rangle$ (predicted in Section 3) into Equation 8.

3. A statistical model for predicting fibre pull-out length

The variable strength (and, consequently, pull-out length) of a brittle fibre like glass or carbon is to do with the distribution of flaws and their size along the fibre length. Experiments show that the strength of such a material is well described by a Weibull distribution equation. On loading a batch of fibres up to a stress, σ, a fraction of them, $P(\sigma)$, will fail; in its simplest form

$$P(\sigma) = 1 - \exp\left[(-\sigma/\sigma_0)^m\right]$$

where m is the Weibull modulus and σ_0 is a characteristic strength.

By taking into account the cumulative failure probability of the fibres with the variation of tensile stress on the fibre along its debonded length, the cumulative probability of the distribution of pull-out lengths can be derived [4]:

$$F(x) = 1 - \frac{\int_0^x P[\sigma(x')]\left(\int_{x'}^{l_d/2}\{1 - P[\sigma(x'')]\}\,dx''\right)dx'}{\int_0^{l_d/2} P[\sigma(x')]\left(\int_{x'}^{l_d/2}\{1 - P[\sigma(x'')]\}\,dx''\right)dx'}$$

where the pull-out length, l_p, is given by

$$l_p = \frac{l_d}{2} - x$$

and $\sigma(x)$ and l_d are given by Equations 3 and 5 respectively.

The probability of a value of l_p between x and $(x + dx)$ is therefore

$$\frac{dF}{dx}dx = \frac{F'(\sigma, x)}{F_1}dx$$

where F_1 is the normalizing factor and $F'(\sigma, x)$ is the differential of the numerator of Equation 9. This distribution assumes that fracture of the fibre is bound to occur within the fibre's debonded length.

When $b > l_d/2$ (Fig. 3), the predicted fibre pull-out length is identical to that in the model for a long fibre [4], since the stress in the fibre is able to exceed the highest possible fibre strength at the matrix crack plane. However, when $b < l_d/2$ the maximum fibre stress never reaches this value. The probability of the fibre breaking rather than pulling out intact is given by

$$F_2 \approx 2P[\sigma(b)]$$

This is twice the cumulative probability of breakage of a fibre when subjected to a stress $\sigma(b)$. The factor of two ensures consistency with the fibre debond length calculation where fibre failure is assumed to occur at the average breaking stress of the fibre. Consequently, if $b = l_d/2$ then $F_2 = 1$, and all fibres fracture before pulling out. In the case where $F_2 < 1$, the phenomenon of "intact fibre pull-out" occurs.

The probability of fracture of a fibre at a distance x from the fibre debond crack tip is therefore

$$f(x)\,dx = \frac{F(x)}{F_1}F_2$$

while the remaining fibres are pulled out intact, the probability being

$$f(b)\,dx = (1 - F_2)\,dx$$

The total probability of fibre pull-out is therefore unity. The fibre pull-out length is

$$l_p = b - x \qquad (b < l_d/2) \tag{10}$$

$$l_p = \frac{l_d}{2} - x \qquad (b > l_d/2) \tag{10a}$$

To allow for the uniform probability of the matrix crack being located at any point along the fibre length, we assume values of b between 0 and $l/2$. For each value of b, i.e. for each crack position, the probability of each fibre pull-out length can be calculated and summed. Finally, the probability of each point is divided by the numerator of crack positions considered, to re-normalize the probability function. The average fibre pull-out length, $\langle l_p \rangle$, then, for substitution into Equation 8, is the length for which the cumulative probability distribution is 0.5.

A computer program was developed to carry out this numerical manipulation [6] (see Appendix A).

4. Predictions of the new model

4.1. Prediction of fibre pull-out length

Fig. 4 shows the predicted average lengths of pulled-out fibres, $\langle l_p \rangle$, for two values of the Weibull modulus m. All other values used in the calculation are typical

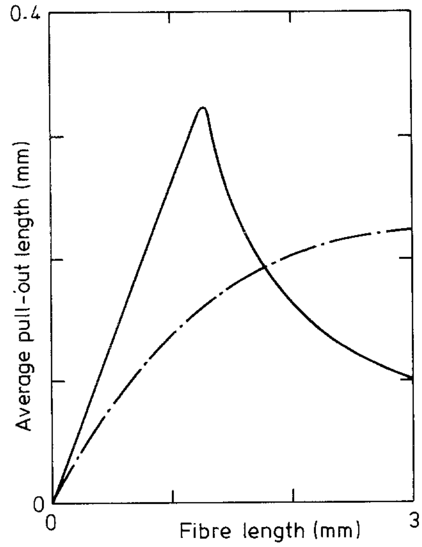

Figure 4 Variation of the average fibre pull-out length, $\langle l_p \rangle$, with fibre length, l, for (---) $m = 7$ and (——) $m = 100$ (Equation 10).

of E-glass fibre–epoxy [4]. First, consider $m = 100$ which corresponds to a fibre of almost uniform strength. The prediction is reminiscent of Kelly's prediction [1], i.e. the intact pulling-out of fibres produces an increasing average pull-out length with increasing length of fibre. But when the fibre length exceeds the fibre debond length (between 1 and 2 mm), only a proportion of fibres are extracted unbroken, the remainder fracturing at the matrix crack plane. However, when $m = 7$, typical of a brittle reinforcing fibre like glass or carbon, the behaviour is quite different. As expected, long fibres exhibit pull-out lengths which are asymptotic to the predicted fibre pull-out length for an infinitely long fibre. But the combination of unbroken pulled-out fibres and those pulling out after snapping away from the matrix crack plane produces a steadily rising average length of pulled out fibre, $\langle l_p \rangle$. There is, therefore, *no* peak fibre pull-out length.

4.2. Prediction of toughness

The maximum toughness of a composite containing aligned, short, brittle fibres randomly dispersed in the matrix can be predicted by summing Equations 6, 7 and 8. A crude allowance for the random orientation of short fibres in a composite can be made by halving the maximum toughness of a similar composite containing randomly dispersed, aligned fibres. (The assumption made is that the fibres produce a two-dimensional random mat and therefore only one-half of them can be considered as aligned parallel to each of the two perpendicular directions.)

Fig. 5 shows the variation of predicted toughness with fibre length for $m = 100$ and $m = 7$, based on Fig. 4. The toughness increases to 95% of ultimate when $l \sim 6l_d$. For fibres of uniform strength ($m = 100$) the behaviour is similar except when $l > l_d$ and the toughness rises more slowly due to a reduction in fibre pull-out energy. A substantial proportion of the energy absorbed is due to the mechanisms involving interfacial debonding (G_d) and fibre fracture (G_f).

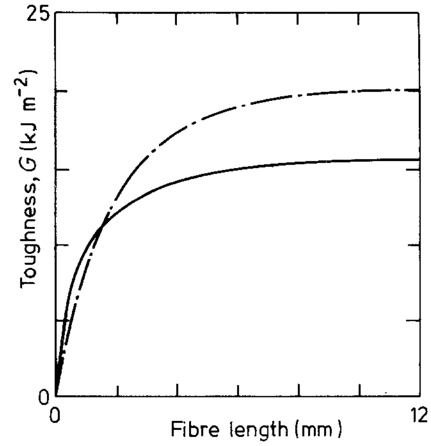

Figure 5 Predicted toughness (summation of Equations 6, 7 and 8 against fibre length for (---) $m = 7$ and (——) $m = 100$.

Consequently, the peak in average fibre pull-out length, $\langle l_p \rangle$, does not appear in the toughness prediction.

The predicted maximum toughness of an aligned two-dimensional glass-fibre–epoxy mat is about $11\,\mathrm{kJ\,m^{-2}}$ (Fig. 5), about a factor 6 or 7 smaller than the predicted value of $72\,\mathrm{kJ\,m^{-2}}$ for a unidirectional fibre–matrix ply [5].

Friedrich [7] has measured the fracture toughness, K_c, of an E-glass fibre–PET composite containing $200\,\mu\mathrm{m}$ long fibres ($V_f = 0.5$), dispersed randomly. Using the relation

$$G_c = K_c^2/E$$

and Friedrich's measured values for the tensile modulus, E, of 20 GPa, and K_c of $8\,\mathrm{MPa\,m^{1/2}}$, we calculate a toughness, G_c, of about $3.2\,\mathrm{kJ\,m^{-2}}$. This is in good agreement with our predicted value (Fig. 5) and supports the validity of our approach.

5. Conclusions

A previous method of calculating fibre pull-out lengths of a broken composite containing short fibres has been reviewed and found to be valid only for the case for which it was originally derived, namely, for a composite containing short fibres uniformly strong. A new model of fibre pull-out for a composite containing short brittle fibres has been shown to be extremely promising. When combined with models of energy-dissipative mechanisms, the new model predicts reasonably accurately the toughness of short-fibre composites. The model predicts a rising toughness with increasing fibre length, finally attaining a constant value, in contrast with the earlier theory.

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Appendix A

The computer program estimates the probability distribution of both intact (unbroken) and fractured pulled-out fibres occurring at many points along the fibre. Having found the average pull-out length, $\langle l_p \rangle$, it then calculates the average toughness of the composite [6].

The program calculates the fibre pull-out distribution using an amended form of Equation 9:

$$F(x) = 1 - \frac{\int_0^x P[\sigma(x')]\,\mathrm{d}x}{\int_0^{l_\mathrm{d}/2} P[\sigma(x')]\,\mathrm{d}x'}$$

The effect of ignoring the probability of survival in more highly stressed regions of the fibre is to slightly alter the shape of $F(x)$ at small x. The effect on the accuracy of the programme is small, and the simplification leads to substantially quicker execution.

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